# Math 612 part 2 - Differential Forms and deRham Cohomology

Note Title 9/15/2014

## Differential Forms on $\mathbb{R}^n$ or open set $U \subset \mathbb{R}^n$.

$x_1, \dots, x_n$ coords on $\mathbb{R}^n$ $\rightsquigarrow$ formal indeterminates $dx_1, \dots, dx_n$, generating a vector space $\mathbb{R}\langle dx_1, \dots, dx_n \rangle = V \cong \mathbb{R}^n$.

$\Omega^* := \Lambda_{\mathbb{R}} V$ : $\mathbb{R}$-alg gen'd by $dx_1, \dots, dx_n$, modulo relations $dx_i \wedge dx_i = 0$, $dx_i \wedge dx_j = -dx_j \wedge dx_i$.

This has $\dim_{\mathbb{R}} = 2^n$, gen'd by $dx_{i_1} \wedge \cdots \wedge dx_{i_k} =: dx_I$

$0 \le k \le n$, $i_1 < i_2 < \cdots < i_k$ $\qquad I = \{i_1, \dots, i_k\}$.

**Def** $U \subset \mathbb{R}^n$ open. $\Omega^*(U) = C^\infty(U) \otimes \Omega^*$.

$\omega \in \Omega^*(U)$, $\omega = \sum_{I \subset \{1, \dots, n\}} f_I \, dx_I$. $\qquad f_I : U \to \mathbb{R}$

$\Omega^*(U)$ is an $\mathbb{R}$-algebra: $\quad (f_I dx_I) \wedge (f_J dx_J) = (f_I f_J)(dx_I \wedge dx_J)$

- graded: $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$, $\quad \Omega^k(U) = \{ \sum_{|I|=k} f_I dx_I \}$.
- sign-commutative: $\omega_1 \in \Omega^{k_1}(U)$, $\omega_2 \in \Omega^{k_2}(U)$
  $\Rightarrow \omega_1 \wedge \omega_2 = (-1)^{k_1 k_2} \omega_2 \wedge \omega_1$.
- $\Omega^0(U) = C^\infty(U)$

**Def** The exterior derivative $d : \Omega^k(U) \to \Omega^{k+1}(U)$ is defined as follows:

$f \in \Omega^0(U) \Rightarrow df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$

$\omega = \sum f_I dx_I \Rightarrow d\omega = \sum df_I \wedge dx_I$.

Note: $d(x_i) = dx_i$, $\quad x_i : U \to \mathbb{R}$.

$|\omega_1| = k \iff \omega_1 \in \Omega^k(U)$.

<u>Prop</u> 1. $d$ is $\mathbb{R}$-linear

2. $d$ is a derivation: $d(\omega_1 \wedge \omega_2) = (d\omega_1)\wedge\omega_2 + (-1)^{|\omega_1|}\omega_1 \wedge d\omega_2$

3. $d$ is a differential: $d^2 = 0$.

<u>Pf</u> 1. obvious.

2. check for $\omega_1 = f_I dx_I$, $\omega_2 = f_J dx_J$.

$$d(\omega_1 \wedge \omega_2) = d(f_I f_J) \wedge dx_I \wedge dx_J = (df_I) f_J \wedge dx_I \wedge dx_J + f_I (df_J) \wedge dx_I \wedge dx_J.$$

3. $d^2(f_I dx_I) = d(df_I \wedge dx_I) = (d^2 f_I) \wedge dx_I - f_I \wedge \underbrace{d(dx_I)}_{0}$

and $$d^2 f = d\left(\sum_i \frac{\partial f}{\partial x_i} dx_i\right) = \sum_{i,j} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j \wedge dx_i$$
$$= \sum_{i<j} \frac{\partial^2 f}{\partial x_j \partial x_i}(dx_j \wedge dx_i - dx_i \wedge dx_j) = 0. \quad \square$$

So we get a complex of $\mathbb{R}$-v.s.'s.

$$0 \longrightarrow \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(U) \longrightarrow 0.$$

<u>Def</u> The <u>deRham cohomology</u> $H^*_{DR}(U)$ is given by

$$H^k_{DR}(U) = \underbrace{\ker(d: \Omega^k U \to \Omega^{k+1} U)}_{\text{closed forms}} \Big/ \underbrace{\operatorname{im}(d: \Omega^{k-1} U \to \Omega^k U)}_{\text{exact forms}}$$

for $0 \le k \le n$; otherwise $0$. Note vector space over $\mathbb{R}$.

<u>Ex</u> 1. $U = \mathbb{R}^0 = pt$. $\Omega^0(pt) = \mathbb{R}$, $H^*_{DR}(pt) = \begin{cases} \mathbb{R}, & * = 0 \\ 0, & * \neq 0 \end{cases}$.

2. $U = \mathbb{R}^1$. $\Omega^0(\mathbb{R}^1) = C^\infty(\mathbb{R})$ $\quad df = \frac{df}{dx}dx \Rightarrow H^0_{DR}(\mathbb{R}) = \mathbb{R}$.

$\Omega^1(\mathbb{R}^1) = \{f\,dx\}$. Any function has an antiderivative $\rightarrow H^1_{DR}(\mathbb{R}) = 0$.

Ex 3. $U \subset \mathbb{R}^3$.

$$\Omega^0 \cong \Omega^3 \quad : \quad \text{functions} \leftrightarrow 0\text{-forms} \leftrightarrow 3\text{-forms}$$

$$\{f : U \to \mathbb{R}\} \qquad \{f\, dx \wedge dy \wedge dz\}$$

vector fields $\leftrightarrow$ 1-forms $\leftrightarrow$ 2-forms

$(f_1, f_2, f_3)$ $\quad f_1 dx + f_2 dy + f_3 dz \quad f_1 dy \wedge dz - f_2 dx \wedge dz + f_3 dx \wedge dy$

this is "Hodge star"

$$\underset{\text{function}}{\Omega^0} \xrightarrow{d} \underset{\text{v.f.}}{\Omega^1} \xrightarrow{d} \underset{\text{v.f.}}{\Omega^2} \xrightarrow{d} \underset{\text{fn.}}{\Omega^3}$$

grad curl div

$d^2 = 0$: gradient fields are irrotational; curl fields are divergence-free.

$H^1(U), H^2(U)$ measure how far the converses are from being true.

## Compactly Supported Cohomology

A function $f$ has compact support if $\operatorname{supp} f = \overline{\{x \in U \mid f(x) \neq 0\}}$ is compact: $\{\text{fns with cpt spt}\} =: C_c^\infty(U)$.

A form $\omega$ has compact support if $\omega = \sum f_I dx_I$, $f_I \in C_c^\infty(U)$.

$$\{\text{forms with cpt spt}\} =: \Omega_c^*(U).$$

Note $d : \Omega_c^k(U) \to \Omega_c^{k+1}(U)$; define $H_c^*(U) = H(\Omega_c^*(U))$ compactly supported cohomology.

Ex $H_c^*(pt) = \begin{cases} \mathbb{R} & * = 0 \\ 0 & \text{else.} \end{cases}$

$H_c^*(\mathbb{R})$? $\qquad 0 \to \Omega_c^0(\mathbb{R}) \xrightarrow{d} \Omega_c^1(\mathbb{R}) \to 0$ $\qquad \underline{H_c^1 \cong \mathbb{R}.}$

$\Uparrow$

$H_c^0$: $\frac{\partial f}{\partial x} = 0 \Rightarrow f = \text{const} \Rightarrow f = 0$.

$H_c^1$: say $f\,dx$, $f \in C_0^\infty(\mathbb{R})$. Then $f\,dx = dg \Leftrightarrow g' = f$; $g \in C_0^\infty(\mathbb{R}) \Leftrightarrow \int f = 0$.

Soon: Poincaré lemma. $H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R} & *=0 \\ 0 & \text{else} \end{cases}$ ; $H^*_c(\mathbb{R}^n) = \begin{cases} \mathbb{R} & *=n \\ 0 & \text{else.} \end{cases}$

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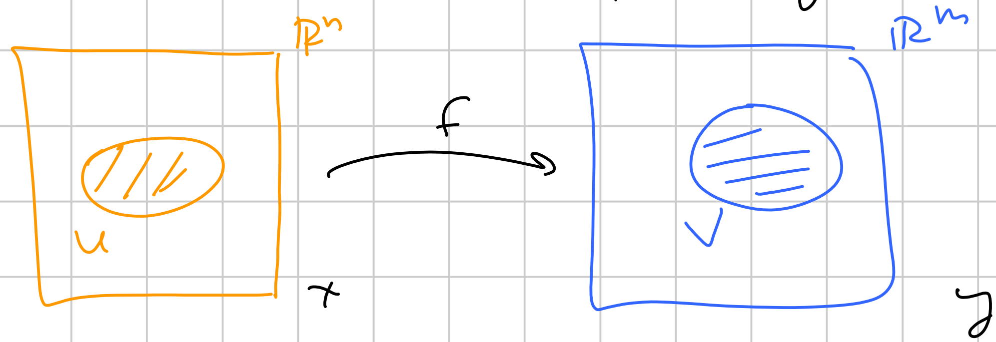
To define cohom. for manifolds, need pullback of forms.




$f = (f_1, \dots, f_m)$, $y_1 = f_1(x_1, \dots, x_n)$ etc.

$f : U \to V$ induces $f^* : C^\infty(V) \to C^\infty(U)$ by $f^*(g) = g \circ f$.
Similarly we get a map

$$f^* : \Omega^k(V) \to \Omega^k(U)$$

$$f^*(g\, dy_{i_1} \wedge \dots \wedge dy_{i_k}) = (g \circ f)\,(df_{i_1} \wedge \dots \wedge df_{i_k}).$$

**Prop** $df^* = f^* d$.

**Pf** $d f^*(g\, dy_{i_1} \wedge \dots \wedge dy_{i_k}) = d(g \circ f) \wedge df_{i_1} \wedge \dots \wedge df_{i_k}$ since $d^2 = 0$

$$f^* d(g\, dy_{i_1} \wedge \dots \wedge dy_{i_k}) = f^*\left(\sum_i \frac{\partial g}{\partial y_i} dy_i \wedge dy_{i_1} \wedge \dots \wedge dy_{i_k}\right)$$
$$= \sum_i \left(\frac{\partial g}{\partial y_i} \circ f\right) df_i \wedge df_{i_1} \wedge \dots \wedge df_{i_k}$$

and

$$d(g \circ f) = \sum_j \frac{\partial (g \circ f)}{\partial x_j} dx_j = \sum_{i,j} \left(\frac{\partial g}{\partial y_i} \circ f\right) \frac{\partial f_i}{\partial x_j} dx_j = \sum_i \left(\frac{\partial g}{\partial y_i} \circ f\right) df_i. \quad \square$$

So: if $f: U \to V$ then $f^*: \Omega^*(V) \to \Omega^*(U)$ induces $f^*: H^k_{DR}(V) \to H^k_{DR}(U)$.

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(U) \arrow[r, "d"] & \Omega^1(U) \arrow[r, "d"] & \cdots \\
0 \arrow[r] & \Omega^0(V) \arrow[r, "d"] \arrow[u, "f^*"] \arrow[ur, phantom, "\circlearrowright"] & \Omega^2(V) \arrow[r, "d"] \arrow[u, "f^*"'] & \cdots
\end{tikzcd}$$

Other useful facts:

<u>Prop</u> 1. $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$

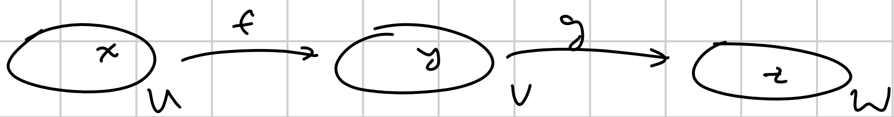
2. $(g \circ f)^* = f^* \circ g^*$.

<u>Pf</u> 1. $f^*((g\, dy_{i_1} \wedge \cdots \wedge dy_{i_k}) \wedge (h\, dy_{j_1} \wedge \cdots \wedge dy_{j_l}))$

$= f^*(gh\, dy_{i_1} \wedge \cdots) = (gh \circ f)\, df_{i_1} \wedge \cdots$

$= (g \circ f)(h \circ f)\, df_{i_1} \wedge \cdots = f^*(g\, dy_{i_1} \wedge \cdots \wedge dy_{i_k}) \wedge f^*(h\, dy_{j_1} \wedge \cdots \wedge dy_{j_l})$.

2.




Because of 1, suffices to check on $h \in C^\infty(W)$ and $dz_i$.

$(g \circ f)^*(h) = h \circ g \circ f = (g^* h) \circ f = f^* \circ g^* \circ h$.

$(g \circ f)^*(dz_i) = d((g \circ f)^* z_i) = d(f^* \circ g^*(z_i)) = f^* \circ g^*(dz_i)$. $\square$

# Poincaré Lemma

<u>Thm</u> $H^*(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & * = 0 \\ 0, & * \neq 0 \end{cases}$ $\quad \forall n \geq 0.$

<u>Pf</u> By induction. True for $n = 0$.

$$\begin{tikzcd} \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}_t \arrow[d, shift left, "\pi"] \\ \mathbb{R}^n_{x_1,\dots,x_n} \arrow[u, shift left, "i"] \end{tikzcd}$$

$i(x_1, \dots, x_n) = (x_1, \dots, x_n, 0)$

$\pi(x_1, \dots, x_n, t) = (x_1, \dots, x_n).$

$$\Rightarrow \begin{tikzcd} \Omega^*(\mathbb{R}^{n+1}) \arrow[d, shift left, "i^*"] \\ \Omega^*(\mathbb{R}^n) \arrow[u, shift left, "\pi^*"] \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} H^*(\mathbb{R}^{n+1}) \arrow[d, shift left, "i^*"] \\ H^*(\mathbb{R}^n) \arrow[u, shift left, "\pi^*"] \end{tikzcd}$$

$\leftarrow$ claim: these are $\cong$ (and in fact inverses).

On chain level: $\pi \circ i = id \Rightarrow i^* \circ \pi^* = id$ on $\Omega^*(\mathbb{R}^n) \Rightarrow$ same on $H^*(\mathbb{R}^n)$.

<u>Claim</u> $\exists\, H : \Omega^k(\mathbb{R}^{n+1}) \longrightarrow \Omega^{k-1}(\mathbb{R}^{n+1})$ s.t.

$$id - \pi^* \circ i^* = (-1)^k (Hd - dH) \text{ on } \Omega^k(\mathbb{R}^{n+1}).$$

Then $H$ is a chain homotopy between $\pi^* \circ i^*$ and $id \Rightarrow \pi^* \circ i^* = id$ on $H^*(\mathbb{R}^{n+1})$.

$$\begin{tikzcd} \cdots \arrow[r] & \Omega^k(\mathbb{R}^{n+1}) \arrow[r, "d"] \arrow[l, bend left, "H"] \arrow[d, shift left, "i^*"] & \Omega^{k+1}(\mathbb{R}^{n+1}) \arrow[r] \arrow[l, bend left, "H"] \arrow[d, shift left, "i^*"] & \cdots \arrow[l, bend left, "H"] \\ \cdots \arrow[r] & \Omega^k(\mathbb{R}^n) \arrow[r, "d"] \arrow[u, shift left, "\pi^*"] & \Omega^{k+1}(\mathbb{R}^n) \arrow[r] \arrow[u, shift left, "\pi^*"] & \cdots \end{tikzcd}$$

<u>Pf</u> $\Omega^k(\mathbb{R}^{n+1})$ gen'd by: $f(x,t)\,dx_I$, $|I| = k$; $f(x,t)\,dx_I \wedge dt$, $|I| = k-1$.

Define:

$$H(f(x,t)\,dx_I) = 0$$

$$H(f(x,t)\,dx_I \wedge dt) = \left( \int_0^t f(x,t)\,dt \right) dx_I.$$

note $i \circ \pi : (x_1, \dots, x_n, t) \mapsto (x_1, \dots, x_n, 0)$ : $(i \circ \pi)^*(dx_j) = d((i \circ \pi)^* x_j) = dx_j$,

$(i \circ \pi)^*(dt) = d((i \circ \pi)^* t) = 0.$

$$(id - \pi^* \circ i^*)(f(x,t)\,dx_I) = f(x,t)\,dx_I - f(x,0)\,dx_I \,;$$

$$(Hd - dH)(f(x,t)\,dx_I) = H\Big(\sum \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_I + \frac{\partial f}{\partial t}\, dt \wedge dx_I\Big)$$

$$= (-1)^k H\Big(\frac{\partial f}{\partial t}\, dx_I \wedge dt\Big)$$

$$= (-1)^k \Big(\int_0^t \frac{\partial f}{\partial t}(x,\tilde t)\, d\tilde t\Big)\, dx_I$$

$$= (-1)^k \big(f(x,t) - f(x,0)\big)\, dx_I \qquad \checkmark$$

$$(id - \pi^* \circ i^*)(f(x,t)\,dx_I \wedge dt) = f(x,t)\,dx_I \wedge dt$$

$$(Hd - dH)(f(x,t)\,dx_I \wedge dt) = H\Big(\sum \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_I \wedge dt\Big) - d\Big(\big(\textstyle\int_0^t f\big)\, dx_I\Big)$$

$$= \sum_i \Big(\int_0^t \frac{\partial f}{\partial x_i}\Big)\, dx_i \wedge dx_I - \sum_i \frac{\partial}{\partial x_i}\Big(\int_0^t f\Big)\, dx_i \wedge dx_I$$

$$- f(x,t)\, dt \wedge dx_I$$

$$= (-1)^k f(x,t)\, dx_I \wedge dt. \qquad \checkmark \qquad \square$$

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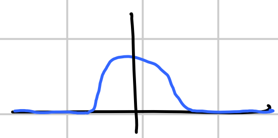
## Poincaré Lemma for compactly supported cohomology

**Thm** $H_c^*(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & * = n \\ 0, & \text{otherwise.} \end{cases}$

Pullback doesn't work: eg if $f(x) =$ 

then $\mathbb{R}^2 \xrightarrow{\pi} \mathbb{R}$, $(x,y) \mapsto x$, $\quad \pi^* f(x,y) = f(x)$ is not compactly supported.

Instead: pushforward (integration along a fiber).

$$\mathbb{R}^n \times \mathbb{R} \xrightarrow{\pi} \mathbb{R}^n \rightsquigarrow \pi_* : \Omega_c^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega_c^{k-1}(\mathbb{R}^n)$$

$$\pi_*(f(x,t)\,dx_I) = 0$$

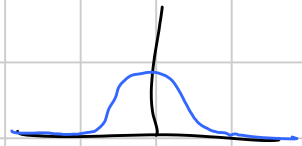
$$\pi_*(f(x,t)\,dx_I \wedge dt) = \Big(\int_{-\infty}^{\infty} f(x,t)\,dt\Big)\, dx_I.$$

Exc: $d\pi_* = \pi_* d$ so $\pi_*$ is a chain map.

$$\begin{tikzcd}
{} \arrow[r] & \Omega_c^k(\mathbb{R}^n \times \mathbb{R}) \arrow[r, "d"] \arrow[dl] & \Omega_c^{k+1}(\mathbb{R}^n \times \mathbb{R}) \arrow[r] \arrow[dl, "\pi_*"'] & {} \arrow[dl] \\
{} \arrow[r] & \Omega_c^k(\mathbb{R}^n) \arrow[r, "d"] & \Omega_c^{k+1}(\mathbb{R}^n) &
\end{tikzcd}$$

$\Rightarrow$ $\pi_*$ induces a map $H_c^k(\mathbb{R}^n \times \mathbb{R}) \to H_c^{k-1}(\mathbb{R}^n)$.

Want this to be an isomorphism.



Let $e \in \Omega_c^1(\mathbb{R})$ be defined by $e = e(t)\,dt$, $\int_{-\infty}^{\infty} e(t)\,dt = 1$.

$\Rightarrow$ get a map

$$e_* : \Omega_c^{k-1}(\mathbb{R}^n) \longrightarrow \Omega_c^k(\mathbb{R}^n \times \mathbb{R})$$
$$\varphi \longmapsto (\pi^*\varphi) \wedge e$$

really $\pi_2^* e$, $\pi_2 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$.

note compact support.

Since $de = 0$, $e_*$ induces a map $H_c^{k-1}(\mathbb{R}^n) \to H_c^k(\mathbb{R}^n \times \mathbb{R})$.

Now:
- $\pi_* e_* = id$
- $\exists\, H : \Omega_c^k(\mathbb{R}^n \times \mathbb{R}) \to \Omega_c^{k-1}(\mathbb{R}^n \times \mathbb{R})$ s.t.

  $id - e_*\pi_* = (-1)^k (Hd - dH)$ in $\Omega_c^k(\mathbb{R}^n \times \mathbb{R})$.

$$H(f(x,t)\,dx_I) = 0$$
$$H(f(x,t)\,dx_I \wedge dt) = \left[\left(\int_{-\infty}^t f(x,t)\,dt\right) - \left(\int_{-\infty}^{\infty} f(x,t)\,dt\right)\left(\int_{-\infty}^t e(t)\,dt\right)\right] dx_I.$$

$$(id - e_*\pi_*)(f\,dx_I) = f\,dx_I$$

$$(-1)^k (Hd - dH)(f\,dx_I) = (-1)^k H\left(\sum \frac{\partial f}{\partial x_i} dx_i \wedge dx_I + (-1)^k \frac{\partial f}{\partial t} dx_I \wedge dt\right)$$
$$= \left[\int_{-\infty}^t \frac{\partial f}{\partial t} - \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial t}\right)\left(\int_{-\infty}^t e\right)\right] dx_I$$
$$= f\,dx_I.$$

$$(id - e_* \pi_*)(f\, dx_I \wedge dt) = f\, dx_I \wedge dt - \left(\int_{-\infty}^{\infty} f\right) e(t)\, dx_I \wedge dt$$

$$\begin{aligned} Hd\,(f\, dx_I \wedge dt) &= H\left( \sum \frac{\partial f}{\partial x_i}\, dx_i \wedge dx_I \wedge dt \right) \\ &= \sum \left( \int_{-\infty}^{t} \frac{\partial f}{\partial x_i} - \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x_i}\right)\left(\int_{-\infty}^{t} e\right) dx_i \wedge dx_I \right) \end{aligned}$$

$$\begin{aligned} dH(f\, dx_I \wedge dt) &= d\left( \left( \int_{-\infty}^{t} f - \left(\int_{-\infty}^{\infty} f\right)\left(\int_{-\infty}^{t} e\right)\right) dx_I \right) \\ &= \sum \left( \int_{-\infty}^{t} \frac{\partial f}{\partial x_i} - \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x_i}\right)\left(\int_{-\infty}^{t} e\right)\right) dx_i \wedge dx_I \\ &\quad + \left(f - \left(\int_{-\infty}^{\infty} f\right) e\right) dt \wedge dx_I \end{aligned}$$

$$\begin{aligned} \Rightarrow (-1)^k (Hd - dH)(f\, dx_I \wedge dt) &= (-1)^{k-1}\left(f - \left(\int_{-\infty}^{\infty} f\right) e\right) dt \wedge dx_I \\ &= \left(f - \left(\int_{-\infty}^{\infty} f\right) e\right) dx_I \wedge dt. \quad \checkmark \end{aligned}$$

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$$\Rightarrow \quad H_c^k(\mathbb{R}^n \times \mathbb{R}) \underset{e_*}{\overset{\pi_*}{\rightleftarrows}} H_c^{k-1}(\mathbb{R}^n) \quad \text{are } \cong. \qquad \square$$

Poincaré: $H_c^n(\mathbb{R}^n) \cong \mathbb{R}$. What's the isomorphism explicitly?

$$H_c^n(\mathbb{R}^n) \xrightarrow{\pi_*} H_c^{n-1}(\mathbb{R}^{n-1}) \xrightarrow{\pi_*} \cdots \xrightarrow{\pi_*} H_c^0(pt) = \mathbb{R}.$$

Each is integration. So the map is

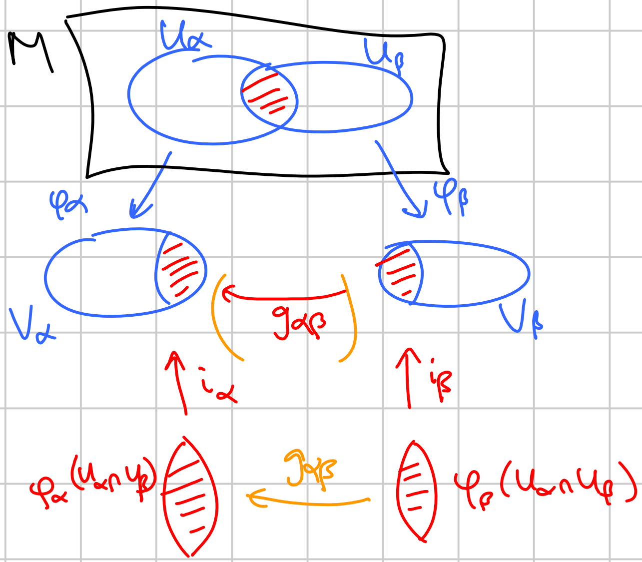
$$\omega \longmapsto \int\!\!\int_{\mathbb{R}^n}\!\cdots\!\int \omega.$$

A generator is (any cpt spt function with $\int = 1$) $dx_1 \wedge \cdots \wedge dx_n$: e.g.

$$e(x_1)\, e(x_2) \cdots e(x_n)\, dx_1 \wedge \cdots \wedge dx_n \longmapsto 1.$$

# Differential Forms on Manifolds

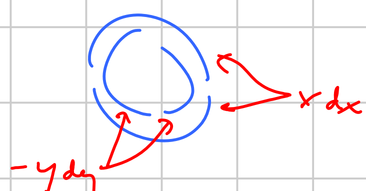
Recall: $M$ = <u>smooth</u> mfd if $\exists$ <u>atlas</u> for $M$:
open cover $\{U_\alpha\}$, $\cup U_\alpha = M$, homeos $\varphi_\alpha : U_\alpha \to V_\alpha \overset{\text{open}}{\subset} \mathbb{R}^n$ s.t.
$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)$ is $\underline{\underline{C^\infty}}$.




(note $g_{\beta\alpha} = g_{\alpha\beta}^{-1}$).

<u>Def</u> A <u>differential form</u> on $M$ is a collection of forms $\omega_\alpha$ on $V_\alpha$ s.t. $\omega_\beta = g_{\alpha\beta}^* \omega_\alpha$ on $\varphi_\beta(U_\alpha \cap U_\beta)$: more precisely,
$$i_\beta^* \omega_\beta = (i_\alpha \circ g_{\alpha\beta})^* \omega_\alpha .$$

<u>Ex</u> $S^1 = \{x^2 + y^2 = 1\}$. 4 charts:




On intersection: $y = \pm\sqrt{1-x^2} \Rightarrow -y\,dy = (\mp\sqrt{1-x^2})\left(\frac{\mp x}{\sqrt{1-x^2}}\,dx\right) = x\,dx$.

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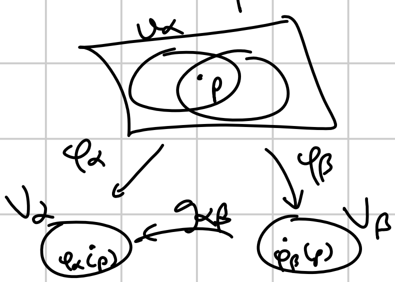
Write $\Omega^*(M) = \{\text{differential forms on } M\}$ $\quad 0 \le * \le n$.

- Since $f^*(\omega \wedge \eta) = (f^*\omega) \wedge (f^*\eta)$, $\wedge$ is well-defined on $\Omega^*(M)$
- Since $df^* = f^*d$, $d$ is well-defined on $\Omega^*(M)$

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \to 0$$

Define $H^k_{DR}(M) = \ker d / \operatorname{im} d$ at $\Omega^k(M)$; $H^*_{DR}(M) = \bigoplus_k H^k_{DR}(M)$.

Similarly we can define compactly supported cohomology. The one point to check: what does it mean for $\omega(p)=0$ when $\omega \in \Omega^*(M)$, $p \in M$?




$\omega_\alpha = \sum f_I \, dx_I \quad f_I : V_\alpha \to \mathbb{R}$

Say $\omega(p)=0 \iff f_I(\varphi_\alpha(p)) = 0 \ \forall I$.

Check indep of coord chart:

$$\omega_\beta = g^*_{\alpha\beta}\omega_\alpha = \sum_I \underbrace{(f_I \circ g_{\alpha\beta})}_{\text{at } \varphi_\beta(p), \text{ get } 0 \text{ if } f_I(\varphi_\alpha(p))=0 \ \forall I.} \big(d(\ ) \wedge d(\ ) \wedge \cdots \wedge d(\ )\big)$$

<u>Def</u> $\Omega^k_c(M) = \{\omega \in \Omega^k(M) \mid \operatorname{supp} \omega = \overline{\{p \mid \omega(p) \neq 0\}} \text{ is cpt}\}$.

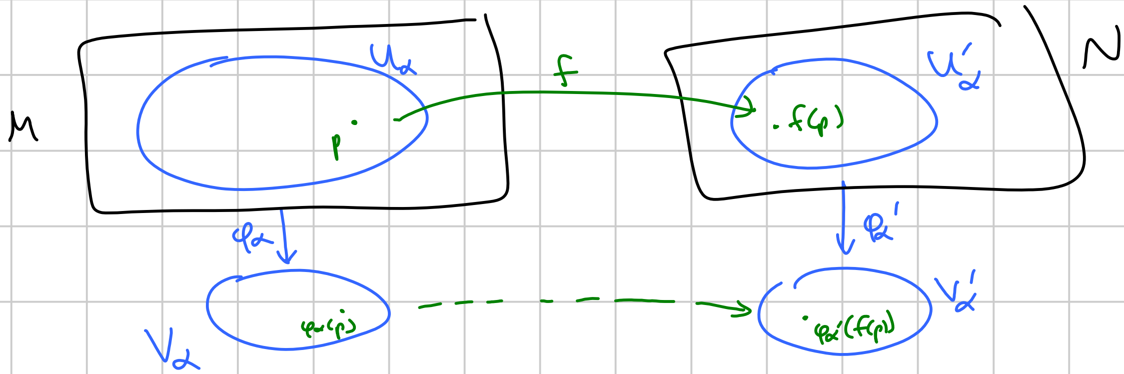
If $\omega = 0$ in a nbd of $p$ then $d\omega = 0$ in the same nbd $\Rightarrow$

$$0 \to \Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_c(M) \to 0$$

$$H^k_c(M) = \ker d / \operatorname{im} d \text{ at } \Omega^k_c(M).$$

# Pullbacks

$M, N$ smooth. A map $f: M \to N$ is smooth if the corresponding maps on coord charts are smooth.




Note we can pull back to get $\Omega^*(V'_\alpha) \to \Omega^*(V_\alpha)$ (or subsets of $V_\alpha, V'_\alpha$).

Thus $f: M \to N$ induces $f^*: \Omega^*(N) \to \Omega^*(M)$ pullback
(check: everything patches together).

Again: $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$, $df^* = f^* d$.

So $f^*$ induces a ring homomorphism $f^*: \Omega^*(N) \to \Omega^*(M)$.

---

Suppose $M$ = smooth mfd, atlas $\{U_\alpha\} \Rightarrow M \times \mathbb{R}$ atlas $\{U_\alpha \times \mathbb{R}\}$.

$$\begin{array}{c} M \times \mathbb{R} \\ \pi \downarrow \uparrow i \\ M \end{array}$$

Same argument as Poincaré Lemma shows that $\pi^*, i^*$ are isomorphisms:

$$H^*(M \times \mathbb{R}) \underset{\pi^*}{\overset{i^*}{\rightleftarrows}} H^*(M).$$

9/23 ↑

**Def** Two smooth maps $f_0, f_1: M \to N$ are homotopic, $f_0 \sim f_1$, if $\exists$ smooth $F: M \times [0,1] \to N$, $F(\cdot, 0) = f_0(\cdot)$, $F(\cdot, 1) = f_1(\cdot)$.

**Prop** $f_0, f_1 : M \to N$ homotopic. Then
$$f_0^* = f_1^* : H^*(N) \longrightarrow H^*(M).$$

**Pf** Extend $F$ to $F : M \times \mathbb{R} \to N$, $F(x,t) = \begin{cases} f_0(x), & t \leq 0 \\ f_1(x), & t \geq 1. \end{cases}$

Define
$$\begin{matrix} i_0 \\ i_1 \end{matrix} : M \hookrightarrow M \times \mathbb{R} \text{ by } \begin{matrix} x \mapsto (x,0) \\ x \mapsto (x,1). \end{matrix}$$

$i_0^*, i_1^*$ both invert $\pi^*$ so $i_0^* = i_1^* : H^*(M \times \mathbb{R}) \to H^*(M)$.

Now
$$M \underset{i_1}{\overset{i_0}{\rightrightarrows}} M \times \mathbb{R} \xrightarrow{F} N \qquad f_0 = F \circ i_0, \ f_1 = F \circ i_1$$

$$\Rightarrow \quad H^*(N) \xrightarrow{F^*} H^*(M \times \mathbb{R}) \underset{i_1^*}{\overset{i_0^*}{\rightrightarrows}} H^*(M)$$

$\Rightarrow f_0^* = i_0^* \circ F^* = i_1^* \circ F^* = f_1^*$. $\square$

**Def** $M, N$ have the <u>same homotopy type</u> if $\exists$ $M \underset{g}{\overset{f}{\rightleftarrows}} N$
with $f \circ g \sim id_N$, $g \circ f \sim id_M$.

<u>ex</u>: $M$ is <u>contractible</u> if $M \sim pt$.

**Cor** If $M, N$ have the same htpy type, then $H^*(M) \cong H^*(N)$.

**Pf**
$$H^*(M) \underset{f^*}{\overset{g^*}{\rightleftarrows}} H^*(N). \quad \square$$

**Ex/Def** $N \subset M$ submanifold, $i : N \to M$.

- $r : M \to N$ is a <u>retraction</u> if $r \circ i = id_N$ (restricts to id on $N$)
- $r$ is a <u>deformation retraction</u>, and $N$ is a <u>deformation retract</u> of $M$, if $i \circ r \sim id_M$.

Deformation retract $\Rightarrow$ same htpy type.

Lame ex: $\mathbb{R}^n$, pt

Better ex: $M = \mathbb{R}^n - \{0\}$, $N = S^{n-1} = \{\|x\| = 1\}$.

$r: M \to N$ is a deformation retract.
$x \mapsto x/\|x\|$

---

Analogous result for cpt spt cohomology:

Note $H_c^*$ isn't an invt of homotopy type.

But: $H_c^*(M \times \mathbb{R})$, $\pi_* \downarrow \uparrow e_*$, $H_c^{*-1}(M)$ — maps can be defined as before, and they're isomorphisms.

---

# Mayer-Vietoris Sequence

First: suppose $N \subset M$ is an open subset

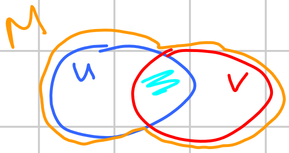
$\Rightarrow$ $i: N \hookrightarrow M$.

Then $i^*: \Omega^*(M) \longrightarrow \Omega^*(N)$

is given by restriction: $\omega \in \Omega^*(M)$,

$i^*\omega = \omega|_N$.

How to compute $H^*$ for something that isn't contractible?
Say $M = U \cup V$, $U, V$ open. If $M = U \amalg V$ then
$H^*(M) = H^*(U) \oplus H^*(V)$.





$U \amalg V$ (think $U \times \{0\} \cup V \times \{1\} \subset M \times \mathbb{R}$)

$$M \longleftarrow U \amalg V \underset{i_V}{\overset{i_U}{\leftleftarrows}} U \cap V$$

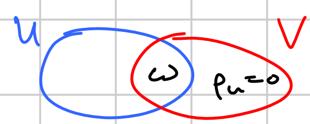
$$\Rightarrow \quad \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \underset{i_V^*}{\overset{i_U^*}{\rightrightarrows}} \Omega^*(U \cap V)$$

<u>Prop</u> The following is exact:
$$0 \longrightarrow \Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{-i_U^* + i_V^*} \Omega^*(U \cap V) \to 0.$$

$$\begin{tikzcd}
 & M & \\
U \arrow[ur] & & V \arrow[ul] \\
 & U \cap V \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

<u>Pf</u> First two steps clear.
Last step: Let $(\rho_U, \rho_V)$ be a partition of unity subordinate to the open cover $\{U, V\}$ i.e. $\operatorname{supp} \rho_U \subset U$, $\operatorname{supp} \rho_V \subset V$, $\rho_U + \rho_V = 1$.
If $\omega \in \Omega^*(U \cap V)$ then

$$\rho_U \cdot \omega \in \Omega^*(V), \quad \rho_V \cdot \omega \in \Omega^*(U)$$




and on $U \cap V$, $\omega = -(-\rho_V \omega) + \rho_U \omega$. $\square$

Recall partitions of unity: A <u>partition of unity</u> on $M$ is a collection $\{\rho_\alpha\}$, $\rho_\alpha : M \to \mathbb{R}$, with

- $\rho_\alpha \geq 0$
- $\forall p \ \exists$ nbd with $\rho_\alpha = 0$ $\forall$ but finitely many $\alpha$
- $\sum \rho_\alpha(p) = 1$.

Let $\{U_\alpha\}$ = open cover for $M$, $M = \cup U_\alpha$.

(a) $\exists$ partition of unity $\{\rho_\alpha\}$ subordinate to $U_\alpha$:
supp $\rho_\alpha \subset U_\alpha$.

(b) $\exists$ partition of unity $\{\rho_\beta\}$ with compact support (index set might be different) s.t. $\forall \beta$, supp $\rho_\beta \subset U_\alpha$ for some $\alpha$.

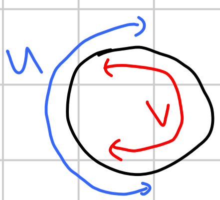
NB can't nec. have both: eg $\{\mathbb{R}\}$ = open cover for $\mathbb{R}$.

---

Consequence: Mayer-Victoris sequence, LES in cohomology

$$\boxed{\begin{array}{c} \to H^{k+1}(M) \to H^{k+1}(U) \oplus H^{k+1}(V) \to H^{k+1}(U\cap V) \to \\ \to H^{k}(M) \to H^{k}(U) \oplus H^{k}(V) \to H^{k}(U\cap V) \to \end{array}}$$

Ex: $S^1$.




$$\begin{array}{lcccc} & S^1 & & U \amalg V & & U\cap V \\ H^1 & & \to & 0 & \to & 0 \\ H^0 & \mathbb{R} & \to & \mathbb{R}\oplus\mathbb{R} & \to & \mathbb{R}\oplus\mathbb{R} \end{array}$$

this map is $(c_1, c_2) \mapsto (c_2 - c_1, c_2 - c_1)$

$\therefore H^1(S^1) = \mathbb{R}$.

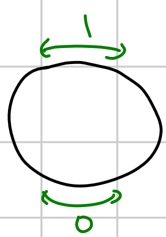
Explicitly: the maps in the LES are clear except coboundary map $\delta : H^k(U\cap V) \to H^{k+1}(M)$.

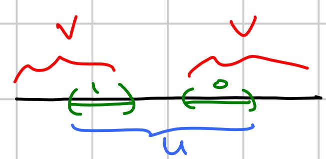
$$\begin{array}{ccccccccc}
0 \to & \Omega^{k+1}(M) & \to & \Omega^{k+1}(U)\oplus\Omega^{k+1}(V) & \to & \Omega^{k+1}(U\cap V) & \to 0 \\
 & \uparrow & & \uparrow & & \uparrow & \\
0 \to & \Omega^{k}(M) & \to & \Omega^{k}(U)\oplus\Omega^{k}(V) & \to & \Omega^{k}(U\cap V) & \to 0
\end{array}$$

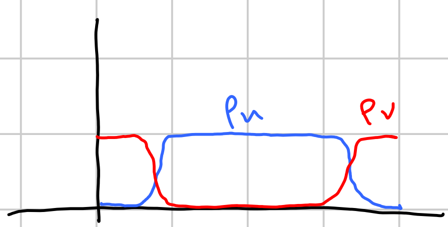
Say $\omega \in \Omega^k(U\cap V)$, $d\omega = 0$. Pull back to $(-\rho_V\omega, \rho_U\omega)$
$\to (-d(\rho_V\omega), d(\rho_U\omega)) \to \delta\omega = \begin{cases} -d(\rho_V\omega) & \text{on } U \\ d(\rho_U\omega) & \text{on } V \end{cases}$

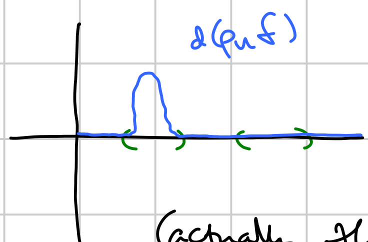
and note on $U\cap V$, these agree since $d(\rho_U\omega)+d(\rho_V\omega) = d\omega = 0$.
Also easy to check: indep of representative $\omega$.

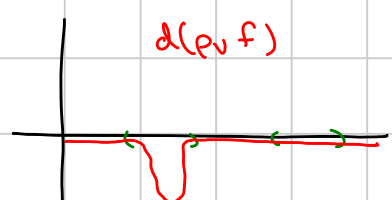
<u>Ex</u> $S^1$. Need $f \in H^0(U\cap V)$ not in image of $H^0(U)\oplus H^0(V)$.
$H^0$ = locally const fns.

$f =$ 

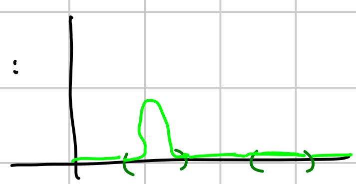

Draw $S^1 = [0,1]/\sim$. 


(actually these times $dt$)

So $\delta f$ is the "bump 1-form" $\tau$:  $dt$.

Note if we write $d(\varphi_U f) = g\,dt$, $g$ supported in $U$, then

$$\int_U g\,dt = \int_U \frac{d}{dt}(\varphi_U f)\,dt = \Delta(\varphi_U f) = 1$$ and $\tau = 0$ outside $U$

so "the integral of $\tau$ over $S^1$ is 1".

9/25 ↑

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## Mayer-Vietoris for Compact Support

Usual MV: $\Omega^*(M) \longrightarrow \Omega^*(U) \oplus \Omega^*(V) \longrightarrow \Omega^*(U \cap V)$.

Maps are restrictions. Now for $\Omega_c^*$, if $i: U \hookrightarrow M$ is an inclusion map of open sets, then $\exists$ "pushforward" $i_*: \Omega_c^*(U) \to \Omega_c^*(M)$,

$$i_*\omega(p) = \begin{cases} \omega, & p \in U \\ 0, & p \notin U \end{cases}.$$

(Point-set exercise: if $A \subset U \subset X$ and the closure of $A$ in $U$ is cpt, then the closure of $A$ in $X$ is cpt.)

Get $\Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longrightarrow \Omega_c^*(M)$.

**Prop** The following is exact:

$$0 \longrightarrow \Omega_c^*(U \cap V) \longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longrightarrow \Omega_c^*(M) \longrightarrow 0$$

$\omega \longmapsto (-i_*\omega, i_*\omega)$

$(\omega, \eta) \longmapsto i_*\omega + i_*\eta$

Pf Exercise. $\square$

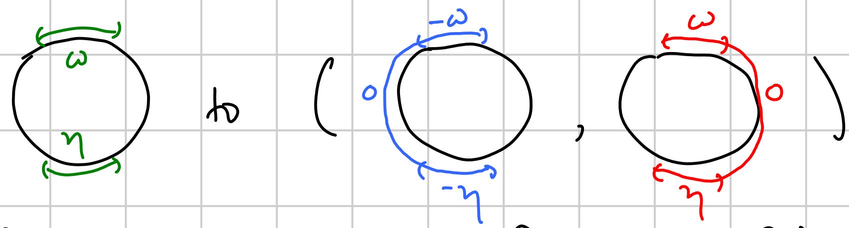
So: get LES, <u>Mayer-Vietoris sequence for cpt spt cohom:</u>

$$\boxed{\begin{array}{c} \cdots \\ H_c^{k+1}(M) \leftarrow H_c^{k+1}(U) \oplus H_c^{k+1}(V) \leftarrow H_c^{k+1}(U \cap V) \\ H_c^{k}(M) \leftarrow H_c^{k}(U) \oplus H_c^{k}(V) \leftarrow H_c^{k}(U \cap V) \\ \cdots \end{array}}$$

<u>Ex</u> $S^1$:

$$\begin{array}{llcll} H_c^2 & & & & 0 \\ H_c^1 & ? & \leftarrow & \mathbb{R}\oplus\mathbb{R} \xleftarrow{\varphi} & \mathbb{R}\oplus\mathbb{R} \\ H_c^0 & ? & \leftarrow & & 0 \end{array}$$

$H_c^1(U) = H_c^1(V) = \mathbb{R}$ (same as $H_c^1(\mathbb{R})$). Map $H_c^1 \xrightarrow[\cong]{\int} \mathbb{R}$.

$\varphi$ maps 



$$\left(\int\omega, \int\eta\right) \longmapsto \left(-\int\omega - \int\eta,\ \int\omega + \int\eta\right).$$

So $\varphi$ = rank 1 $\rightarrow$ $H_c^0(S^1) = H_c^1(S^1) = \mathbb{R}$

# Integration/Orientation

$M$ smooth mfd, atlas $\{(U_\alpha, \varphi_\alpha)\}$.

$g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ is orientation-preserving if the Jacobian $Dg_{\alpha\beta}$ has det $> 0$.

**Def** $M$ is orientable if it has an oriented atlas: one for which all $g_{\alpha\beta}$'s are orientation-preserving.

**Prop** $M$ orientable $n$-mfd $\Leftrightarrow$ it has a nowhere vanishing $n$-form $\omega$.

**Lemma** $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ diffeo, $\varphi(y_1, \dots, y_n) = (x_1, \dots, x_n)$. Then

$$\varphi^*(dx_1 \wedge \cdots \wedge dx_n) = \det(D\varphi)\, dy_1 \wedge \cdots \wedge dy_n.$$

**Pf of Prop**

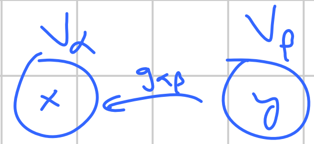
$\boxed{\Leftarrow}$: Let $(U_\alpha, \varphi_\alpha)$ be a chart, $U_\alpha$ connected, $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$. On $U_\alpha$,

$$\omega_\alpha = f_\alpha\, dx_1 \wedge \cdots \wedge dx_n \quad , \quad f \text{ nowhere zero.}$$

Either $f_\alpha > 0$ or $f_\alpha < 0$. If $f_\alpha > 0$, fine, if $f_\alpha < 0$, replace $\varphi_\alpha$ by $T \circ \varphi_\alpha$, $T(x_1, \dots, x_n) = (-x_1, x_2, \dots, x_n)$.

$$T^*(f_\alpha dx_1 \wedge \cdots \wedge dx_n) = \overbrace{-(f_\alpha \circ T)}^{>0}\, dx_1 \wedge \cdots \wedge dx_n$$

$\Rightarrow$ Can assume all $f_\alpha > 0$. Then:




$$\begin{aligned} f_\beta\, dy_1 \wedge \cdots \wedge dy_n = \omega_\beta = g_{\alpha\beta}^* \omega_\alpha &= g_{\alpha\beta}^*(f_\alpha dx_1 \wedge \cdots \wedge dx_n) \\ &= (f_\alpha \circ g_{\alpha\beta})\, dx_1 \wedge \cdots \wedge dx_n \quad \leftarrow x_i = x_i(y_1, \dots, y_n) \\ &= (f_\alpha \circ g_{\alpha\beta}) \det(Dg_{\alpha\beta})\, dy_1 \wedge \cdots \wedge dy_n \end{aligned}$$

$\Rightarrow \underline{\det(Dg_{\alpha\beta}) > 0}$.

$\boxed{\Rightarrow}$: Start with an oriented atlas $\{(U_\alpha, \varphi_\alpha)\}$. Define $\omega_\alpha$ on $U_\alpha$ by
$\omega_\alpha = \rho_\alpha \, dx_1 \wedge \cdots \wedge dx_n$ for $\rho_\alpha$ = part. of unity subordinate to $\{U_\alpha\}$.
Extend $\omega_\alpha$ to all of $M$ by $0$ outside $U_\alpha$.

Now write $\omega = \sum_\alpha \omega_\alpha$. At each point, this is
(finite positive sum) $dx_1 \wedge \cdots \wedge dx_n$

So nowhere zero. $\square$

<u>Def</u> A <u>volume form</u> is a nowhere vanishing $n$-form.
Note two volume forms $\omega, \omega'$ satisfy $\omega' = f\omega$ for some $f \in C^\infty(M)$ nowhere zero. Say two volume forms are <u>equivalent</u> if $f > 0$ everywhere.
An <u>orientation</u> of $M$ is a choice of equiv. class of volume forms.

Note: $M$ connected orientable $\Rightarrow$ $\exists$ two orientations.
ex: $M = \mathbb{R}^n$, $\quad dx_1 \wedge \cdots \wedge dx_n$, $\; -dx_1 \wedge \cdots \wedge dx_n$.

<u>Integration</u> If $U \subset \mathbb{R}^n$ and $\omega \in \Omega^n_c(U)$, $\omega = f \, dx_1 \wedge \cdots \wedge dx_n$,
then define $\quad \int_U \omega = \int_U f \, dx_1 \cdots dx_n$.
How does this change under diffeo?

Say $\varphi : V \to U$ diffeo, $x = \varphi(y)$.

$$\underset{V \subset \mathbb{R}^n}{y} \xrightarrow{\varphi} \underset{U \subset \mathbb{R}^n}{x}$$

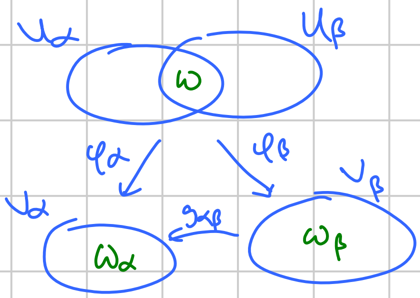
$$\varphi^* \omega = (f \circ \varphi) \det(D\varphi) \, dy_1 \wedge \cdots \wedge dy_n .$$

Change of variables formula:

depends on orient-preserving or orient-reversing.

$$\int_V \varphi^* \omega = \int_V (f \circ \varphi) \det(D\varphi) \, dy_1 \cdots dy_n = \pm \int_U f \, dx_1 \cdots dx_n = \pm \int_U \omega .$$

So if we stick to orient-preserving, then $\int$ makes sense indep of coord chart:




if $\operatorname{supp}\omega \subset U_\alpha \cap U_\beta$ then

$$\omega_\beta = g_{\alpha\beta}^* \omega_\alpha ,$$

$$\int_{V_\alpha} \omega_\alpha = \int_{V_\beta} \omega_\beta$$ ← we can call either of these $\int_M \omega$.

<u>Def</u> $M$, $(U_\alpha, \varphi_\alpha)$ oriented atlas, $\rho_\alpha$ = subordinate partition of unity. If $\omega \in \Omega^n_c(M)$, define

$$\int_M \omega = \sum \int_{V_\alpha} \rho_\alpha \omega_\alpha \qquad \text{(assume sum is finite).}$$

↑ here "$\rho_\alpha$" $= \rho_\alpha \circ \varphi_\alpha^{-1} : V_\alpha \to \mathbb{R}$, $\rho_\alpha \omega_\alpha \in \Omega^n_c(V_\alpha)$.

<u>Note</u> depends on orientation! If we take the other one, $\int_M \omega \to -\int_M \omega$.

<u>Prop</u> $\int_M \omega$ is indep of atlas, partition of unity.

<u>Pf</u> Say we have two atlases $(U_\alpha, V_\alpha, \varphi_\alpha, \rho_\alpha)$, $(\tilde{U}_\beta, \tilde{V}_\beta, \tilde{\varphi}_\beta, \tilde{\rho}_\beta)$.
Then

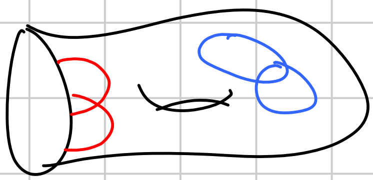
$$\sum_\alpha \int_{V_\alpha} \rho_\alpha \omega_\alpha = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_{\alpha\beta} \int_{U_\alpha} \rho_\alpha \tilde{\rho}_\beta \omega = \sum_{\alpha,\beta} \int_{U_\alpha \cap \tilde{U}_\beta} \rho_\alpha \tilde{\rho}_\beta \omega$$

$$= \cdots = \sum_\beta \int_{\tilde{V}_\beta} \tilde{\rho}_\beta \tilde{\omega}_\beta . \qquad \square$$

# Manifolds with boundary

From now on, we'll stipulate that all $U_\alpha$ are diffeomorphic to $\mathbb{R}^n$ (in particular, connected).

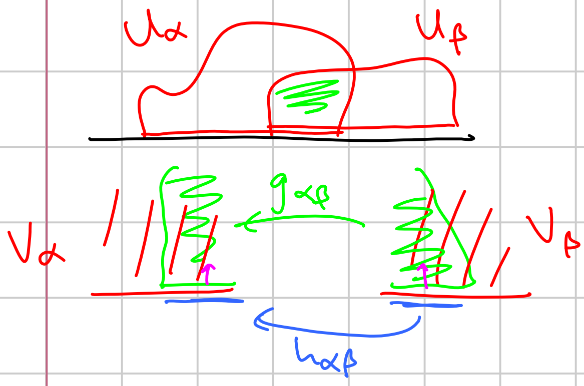
Def $M$ is an $n$-dimensional manifold with boundary if $M$ is covered by charts $U_\alpha$ s.t. each $U_\alpha \cong \mathbb{R}^n$ or upper half space $\{x_n \geq 0\} \subset \mathbb{R}^n$.



$M$ The boundary $\partial M$ of $M$ is then a closed (= no bdy) $(n-1)$-dim mfd, atlas given by $\partial U_\alpha$.

9/30

An oriented atlas for $M$ induces an oriented atlas for $\partial M$:



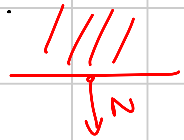

Say $\det Dg_{\alpha\beta} > 0$. Then

$$Dg_{\alpha\beta} = \left(\begin{array}{c|c} Dh_{\alpha\beta} & * \\ \hline 0 & + \end{array}\right)$$

So $\det Dh_{\alpha\beta} > 0$ as well.

As before, an orientation on $M$ is locally $\pm dx_1 \wedge \cdots \wedge dx_n$.
We have a choice about which orientation this induces on $\partial M$.
Choose: $\{x_n \geq 0\}$ $dx_1 \wedge \cdots \wedge dx_n \longrightarrow (-1)^n dx_1 \wedge \cdots \wedge dx_{n-1}$. $\{x_n = 0\}$

Mnemonic: "outward normal first"

Let $N$ = outward normal,  $N = -\frac{\partial}{\partial x_n}$.

Induced orientation is

$$i_N(dx_1 \wedge \cdots \wedge dx_n) = i_N\left((-1)^{n-1} dx_n \wedge dx_1 \wedge \cdots\right) = (-1)^n dx_1 \wedge \cdots$$

i.e. $\underline{i_N \omega_M = \omega_{\partial M}}$. Write: $[M]$ induces $[\partial M]$.

On $M$, can integrate $n$-forms as usual.

<u>Stokes' Thm</u> $M^n$ with $\partial$, oriented, $\omega \in \Omega_c^{n-1}(M)$.

Then $\omega$ pulls back to $\omega \in \Omega_c^{n-1}(\partial M)$ (cpt spt as well) and

$$\boxed{\int_M d\omega = \int_{\partial M} \omega}.$$

<u>Pf</u> HW: prove cases $M = \mathbb{R}^n$ $(\partial M = \emptyset)$ and $M = \{x_n \geq 0\}$ $(\partial M = \{x_n = 0\})$.

Then if $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, $\rho_\alpha \omega$ has cpt spt in $U_\alpha$ so

$$\int_{U_\alpha} d(\rho_\alpha \omega) = \int_{\partial U_\alpha} \rho_\alpha \omega.$$

$$\Rightarrow \int_M d\omega = \sum_\alpha \int_M (\rho_\alpha d\omega) = \sum_\alpha \int_M \left(d(\rho_\alpha \omega) - (d\rho_\alpha) \wedge \omega\right)$$

$$= \sum_\alpha \int_{U_\alpha} d(\rho_\alpha \omega) - \int_M \underbrace{\left(\sum_\alpha d\rho_\alpha\right)}_{0 \text{ since } \sum \rho_\alpha = 1} \wedge \omega$$

$$= \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega = \int_{\partial M} \omega. \qquad \square$$

Significance of Stokes: the map

$$\Omega^k(M) \longrightarrow C^k_{sing}(M;\mathbb{R})$$
$$\omega \longmapsto \left(\sigma \mapsto \int_\sigma \omega\right)$$

if $\sigma: \Delta_k \to M$ then $\int_\sigma \omega := \int_{\Delta_k} \sigma^* \omega$

descends to

$$H^k_{DR}(M) \longrightarrow H^k_{sing}(M;\mathbb{R}):$$

if $\sigma = \partial\sigma'$ then $\int_{\partial\sigma'} \omega = \int_{\sigma'} d\omega = 0$

if $\omega = d\omega'$ then $\int_\sigma \omega = \int_{\partial\sigma} \omega' = 0$.

<u>DeRham Thm</u> The map $H^*_{DR}(M) \to H^*_{sing}(M;\mathbb{R})$ is an <u>isomorphism</u>.

(We'll prove by establishing $\cong$ of both with Čech cohomology.)

Another important thing: if $d\omega$ = exact, $M$ orientable <u>compact</u> <u>closed</u>, then
$$\int_M d\omega = 0$$
so $\int_M$ descends to a map $H^n_{DR}(M) \to \mathbb{R}$.

Additionally, if $\omega$ = vol. form on $M$, $\omega = f\, dx_1 \wedge \cdots \wedge dx_n$ locally, then
$$\int_M \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\alpha \int_{U_\alpha} \rho_\alpha f\, dx_1 \cdots dx_n > 0$$
So
$$\int_M : H^n_{DR}(M) \twoheadrightarrow \mathbb{R}.$$
In fact, we'll see (Poincaré duality) if $M$ connected, then
$$\int_M : H^n_{DR}(M) \xrightarrow{\cong} \mathbb{R}.$$

Can use this to find a generator of $H^n(M)$: e.g. on $S^1$, $\int d\theta = 2\pi$.

Next: use integration + MV to prove Poincaré duality.
Need our atlases to be better-behaved than up to now.

## Good Covers

<u>Def</u> $\{U_\alpha\}$ open cover of $M$ is a <u>good cover</u> if all finite intersections $U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}$ are either $\emptyset$ or diffeo to $\mathbb{R}^n$.
(in particular, all $U_{\alpha_i} \cong \mathbb{R}^n$).

<u>Prop</u> Any smooth manifold has a good cover, finite if cpt.

<u>Pf</u> Choose a Riem metric on $M$. A <u>geodesically convex</u> set in $M$ is one st. any two pts have a unique minimal-length geodesic between them, and this geodesic lies <u>in the set</u>.
Every suff. small geodesically convex open set is $\cong \mathbb{R}^n$.

Any point has a geod. convex nbd, and intersections of geod. convex sets are still geod. convex. $\square$

<u>Def</u> $\{U_\alpha\}_{\alpha\in I}$, $\{V_\beta\}_{\beta\in J}$ open covers of $M$. $\{V_\beta\}$ is a <u>refinement</u> of $U_\alpha$ if $\exists$ map $\varphi: J \to I$ st. $V_\beta \subset U_{\varphi(\beta)}$ $\forall \beta$.

**Prop** Any open cover of $M$ has a good refinement.

**Pf** Shrink the geod convex nbds to lie inside $U_\alpha$'s. $\square$

**Prop** If $M$ has a finite good cover, then $H^*_{DR}(M)$, $H^*_c(M)$ are finite-dimensional.

**Pf** We'll do $H^*_{DR}(M)$. MV:

$$\cdots \longrightarrow H^{k-1}(U\cap V) \xrightarrow{\ \delta\ } H^k(U\cup V) \xrightarrow{\ r\ } H^k(U)\oplus H^k(V) \longrightarrow \cdots$$

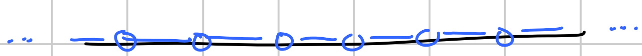
$H^k(U\cup V) \cong \operatorname{im}\delta \oplus \operatorname{im} r$ so if $H^*(U), H^*(V), H^*(U\cap V)$ are finite-dim then so is $H^*(U\cup V)$.

Now induct on # of open sets. $M$ has good cover $\{U_0, \dots, U_m\}$

$\Rightarrow$ $M' = U_0 \cup \dots \cup U_{m-1}$ has good cover $\{U_0, \dots, U_{m-1}\}$

and $M' \cap U_m$ has good cover $\{U_0\cap U_m, \dots, U_{m-1}\cap U_m\}$. $\square$

Note: Not true without finiteness condition: $M = \mathbb{Z}$ or



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# Poincaré Duality

M orientable, finite good cover. Recall $\int_M : H^n_c(M) \to \mathbb{R}$.
This induces $\int : H^k(M) \otimes H_c^{n-k}(M) \longrightarrow H^n_c(M) \longrightarrow \mathbb{R}$

$$[\varphi] , [\psi] \longmapsto [\varphi \wedge \psi] \longmapsto \int_M \varphi \wedge \psi .$$

note well-defined

This is a pairing $\langle , \rangle$ of vector spaces $H^k(M)$, $H^{n-k}_c(M)$.

<u>Prop</u> $\langle , \rangle : V \otimes W \to \mathbb{R}$, $V, W$ fin-dim v.s./$\mathbb{R}$. TFAE:

① it induces $V \xrightarrow{\cong} W^*$

② it induces $W \xrightarrow{\cong} V^*$

③ if $\langle v, \cdot \rangle = 0$ then $v = 0$ <u>and</u> if $\langle \cdot, w \rangle = 0$ then $w = 0$.

In all cases we say $\langle , \rangle$ is <u>nondegenerate</u>.

<u>Pf</u> ①+② ⇒ ③ : obvious.

③ ⇒ ① : injections $V \hookrightarrow W^*$, $W \hookrightarrow V^*$; now count dimension.

① ⇒ ② : $V \xrightarrow{\cong} W^*$ has dual map $W \xrightarrow{\cong} V^*$. □

<u>Poincaré duality</u> M orientable, finite good cover. Then

$$\int : H^k(M) \otimes H^{n-k}_c(M) \to \mathbb{R}$$ is a nondegenerate pairing.

<u>Pf</u> Write $\int : H^k(M) \to (H^{n-k}_c(M))^*$. Suppose $M = U \cup V$.
Write down LES for $H^*$, and dual to LES for $H^*_c$:

$$\begin{array}{ccccccc} \cdots \to H^k(U\cup V) & \longrightarrow & H^k(U)\oplus H^k(V) & \longrightarrow & H^k(U\cap V) & \longrightarrow & H^{k+1}(U\cup V) \to \cdots \\ \downarrow S_M & & \downarrow S_U \oplus S_V & & \downarrow S_{U\cap V} & & \downarrow S_M \\ \cdots \to H_c^{n-k}(U\cup V)^* & \longrightarrow & H_c^{n-k}(U)^* \oplus H_c^{n-k}(V)^* & \longrightarrow & H_c^{n-k}(U\cap V)^* & \longrightarrow & H_c^{n-k-1}(U\cup V)^* \to \cdots \end{array}$$

<u>Claim:</u> $\int$ is a chain map.

Then: by Five Lemma, if PD holds for $U\cap V$, $U$, $V$ ($\downarrow$ are $\cong$), then PD holds for $U\cup V$.

Now induct on # of sets in cover.

$M = U_0 \cup \cdots \cup U_m$. If $m=0$ then $M=\mathbb{R}^n$ and PD holds there: $\int : H_c^n(\mathbb{R}^n) \cong \mathbb{R}$.

otherwise $M' = U_0 \cup \cdots \cup U_{m-1}$; holds for $M'$, $U_m = \mathbb{R}^n$, and $M' \cap U_m = (U_0 \cap U_m) \cup \cdots \cup (U_{m-1}\cap U_m)$, done.

<u>Pf of claim</u> First two squares commute by arrow-chasing.

Last square:

$$\begin{array}{ccc} H^k(U\cap V) & \xrightarrow{\delta} & H^{k+1}(U\cup V) \\ \int \downarrow & & \int \downarrow \\ H_c^{n-k}(U\cap V)^* & \xrightarrow{(-1)^{k+1}\tilde{\delta}^*} & H_c^{n-k-1}(U\cup V)^* \end{array}$$

$(\tilde{\delta}: H_c^{n-k-1}(U\cup V) \to H_c^{n-k}(U\cap V))$

Recall defs of $\delta, \tilde{\delta}$.

<u>$\delta$</u>: if $[\omega] \in H^k(U\cap V)$ ($\omega \in \Omega^k$, $d\omega = 0$) then $\delta\omega = \begin{cases} -d(\rho_V\omega) & \text{on } U \\ d(\rho_U\omega) & \text{on } V \end{cases}$

(note: actually supported in $U\cap V$).

<u>$\tilde{\delta}$</u>: we didn't do this explicitly, so let's do that.

$$0 \longrightarrow \Omega_c^*(U\cap V) \longrightarrow \Omega_c^*(U) \oplus \Omega_c^*(V) \longrightarrow \Omega_c^*(U\cup V) \longrightarrow 0$$

$$\omega \longmapsto (-i_*\omega, i_*\omega)$$

$$(\eta, \tau) \longmapsto i_*\eta + i_*\tau.$$

$$\begin{tikzcd}[column sep=small]
 & -d(\rho_U\tau)=d(\rho_V\tau) \arrow[r] & (d(\rho_U\tau),\ d(\rho_V\tau)) & & \\
0 \arrow[r] & \Omega_c^{n-k}(U\cap V) \arrow[r] & \Omega_c^{n-k}(U)\oplus\Omega_c^{n-k}(V) \arrow[r] & {} & \\
 & {} \arrow[u] & {} \arrow[u, "d"] & {} \arrow[u] & \\
 & {} \arrow[r] & \Omega_c^{n-k-1}(U)\oplus\Omega_c^{n-k-1}(V) \arrow[r] \arrow[uu, bend left] & \Omega_c^{n-k-1}(U\cup V) \arrow[r] & 0 \\
 & & (\rho_U\tau,\rho_V\tau) & \tau \arrow[l, dashed] &
\end{tikzcd}$$

So $\tilde{\delta}(\tau) = -d\rho_U \wedge \tau$.

Now go around the square:

$$\begin{tikzcd}
\omega \arrow[r] \arrow[d] & \delta\omega \arrow[d] \\
\left(\eta \mapsto \int_{U\cap V} \omega\wedge\eta\right) \arrow[r] & \tau \mapsto \int_{U\cup V} (\delta\omega)\wedge\tau = \int_{U\cap V} d(\rho_U\omega)\wedge\tau = \int_{U\cap V} (d\rho_U)\wedge\omega\wedge\tau
\end{tikzcd}$$

$$\tau \mapsto (-1)^{k+1}\int_{U\cap V} \omega\wedge\tilde{\delta}\tau = (-1)^k \int_{U\cap V} \omega\wedge d\rho_U\wedge\tau = \int_{U\cap V} (d\rho_U)\wedge\omega\wedge\tau.$$

$\square$

**Remarks**

1. In fact, don't need finite good cover:

<u>General PD</u> $M$ orientable, $\dim n$. Then
$$\int : H^k(M) \longrightarrow (H_c^{n-k}(M))^* \quad \text{is } \cong.$$

<u>Note</u> <u>Not</u> true for $H_c^{n-k}(M) \longrightarrow (H^k(M))^*$!

Ex: $M = \mathbb{Z}$. $\quad H_c^0(M) = \oplus \mathbb{R}$ direct sum

$\quad H^0(M) = \prod \mathbb{R}$ direct product

and $(\oplus\mathbb{R})^* \cong \prod\mathbb{R}$ but $(\prod\mathbb{R})^* \not\cong \oplus\mathbb{R}$.

2. <u>Cor</u> $M$ connected, oriented $\Rightarrow$ $H_c^n(M) \cong \mathbb{R}$.

If $M$ compact $\Rightarrow$ $H^n(M) \cong \mathbb{R}$.

Digression: <u>degree</u>.

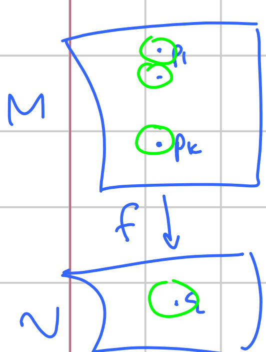
$f: M \to N$ $M, N$ connected cpt oriented mfds of <u>same</u> dim $n$.
Then
$$\begin{array}{ccc} f^*: H^n(N) & \longrightarrow & H^n(M) \\ \int_N \downarrow \cong & & \int_M \downarrow \cong \end{array}$$
induces $\mathbb{R} \xrightarrow{f^*} \mathbb{R}$:
That is, if $\int_N \omega = 1$ then $\int_M f^*\omega = d \in \mathbb{R}$
and $d$ is the <u>degree</u> of $f$.

<u>Prop</u> $d \in \mathbb{Z}$.
(This is surprising here! Maybe not so surprising if we define $d$ to be the map
$$\begin{array}{ccc} H^n(N;\mathbb{Z}) & \longrightarrow & H^n(M;\mathbb{Z}) \\ \cong & & \cong \\ \mathbb{Z} & (PD) & \mathbb{Z} \end{array}$$
: but unclear these are the same.)

<u>Rmk</u> Works equally well for noncompact, if $f: M \to N$ is <u>proper</u>. ($f^{-1}(\text{cpt}) = \text{cpt}$)
$f^*: H^n_c(N) \to H^n_c(M)$

<u>Outline of proof</u>.
Say $q \in N$ is a <u>critical value</u> of $f$ if $\exists\, p \in M$ with $f(p) = q$ and $Df = df_p$ ($n \times n$ matrix) isn't an isomorphism; <u>regular value</u> otherwise.
<u>Sard's Thm</u>: Critical values have measure 0 in $N$.

Pick a regular value $q$. Then $f^{-1}(q) = \text{finite set}\{p_1, \dots, p_k\}$
and $f$ is a local diffeo in a nbd of each $p_i$.




Choose a bump form $\omega \in \Omega^n(N)$ supported in a nbd of $q$, with $\int \omega = 1$. Then $f^*\omega$ = bump forms supported in nbds of $p_1, \dots, p_k$. In each nbd, $\int f^*\omega = \pm \int \omega = \pm 1$ (diffeos preserve $\int$, up to sign). So
$\int_M f^*\omega$ = # of inverse images of $q$, counted with sign. $\square$

# Künneth Formula

$M, N$ smooth mfds.

$$\begin{array}{ccc} & M\times N & \\ {}^{\pi_1}\swarrow & & \searrow^{\pi_2} \\ M & & N \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \Omega^*(M)\otimes\Omega^*(N) & \longrightarrow & \Omega^*(M\times N) \\ \omega\otimes\eta & \longmapsto & \pi_1^*\omega\wedge\pi_2^*\eta \end{array}$$

$\rightsquigarrow$ get a map, <u>cross product</u>

$$\Psi : H^*(M)\otimes H^*(N) \longrightarrow H^*(M\times N).$$

<u>Künneth Thm</u> If $M$ has a finite good cover, then this map is an isomorphism.

<u>Pf</u> Say $\dim M = m$, $\dim N = n$. If $M = \mathbb{R}^m$ then true by Poincaré Lemma.
Want: if true for $M = U$ and $M = V$ and $M = U\cap V$, then true for $M = U\cup V$.

M-V: $\quad \cdots \to H^k(U\cup V) \to H^k(U)\oplus H^k(V) \to H^k(U\cap V) \to H^{k+1}(U\cup V) \to \cdots$

$\otimes$ with $H^{\ell-k}(N)$ (preserves exactness), then $\bigoplus k=0,\dots,n$.

Get the top row of the diagram on the next page.
Want the diagram to be commutative; then done by Five Lemma.

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \bigoplus_k \left(H^k(U\cup V)\otimes H^{\ell-k}(N)\right) \arrow[r] \arrow[d, "\psi"] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_k \left(H^k(U)\otimes H^{\ell-k}(N)\right) \oplus \left(H^k(V)\otimes H^{\ell-k}(N)\right) \arrow[r] \arrow[d, "\psi\oplus\psi"] \arrow[dr, phantom, "\circlearrowright"] & \bigoplus_k H^k(U\cap V)\otimes H^{\ell-k}(N) \arrow[r, "\bigoplus_k(\delta\otimes \mathrm{id})"] \arrow[d, "\psi"] \arrow[dr, phantom, "\text{tricky!}\ \circlearrowright"] & \bigoplus_k H^{k+1}(U\cup V)\otimes H^{\ell-k}(N) \arrow[r] \arrow[d, "\psi"] & \cdots \\
\cdots \arrow[r] & H^\ell((U\cup V)\times N) \arrow[r] & H^\ell(U\times N)\oplus H^\ell(V\times N) \arrow[r] & H^\ell((U\cap V)\times N) \arrow[r, "\delta"] & H^{\ell+1}((U\cup V)\times N) \arrow[r] & \cdots
\end{tikzcd}$$

Say $[\omega]\in H^k(U\cap V)$, $[\eta]\in H^{\ell-k}(N)$.

$$\psi\circ(\delta\otimes \mathrm{id})(\omega\otimes\eta) = \psi(\delta\omega\otimes\eta) = \pi_1^*\delta\omega\wedge\pi_2^*\eta$$
$$\delta\psi(\omega\otimes\eta) = \delta(\pi_1^*\omega\wedge\pi_2^*\eta).$$

$\{\rho_U,\rho_V\}$ partition of unity on $U\cup V$
$\rightsquigarrow \{\pi_1^*\rho_U, \pi_1^*\rho_V\}$ on $(U\cup V)\times N$

$$\delta(\pi_1^*\omega\wedge\pi_2^*\eta) = \begin{cases} -d\big(\overbrace{(\pi_1^*\rho_V)\,\pi_1^*\omega}^{\pi_1^*(\rho_V\omega)}\wedge\pi_2^*\eta\big) & \text{on } U\times N \\ d\big(\underbrace{(\pi_1^*\rho_U)\,\pi_1^*\omega}_{\pi_1^*(\rho_U\omega)}\wedge\pi_2^*\eta\big) & \text{on } V\times N \end{cases}$$

$$= \begin{cases} -d(\pi_1^*(\rho_V\omega))\wedge\pi_2^*\eta \pm \pi_1^*(\rho_V\omega)\,\cancelto{0 \text{ since } d\eta=0}{d\pi_2^*\eta} & \text{on } U\times N \\ \cdots & \text{on } V\times N \end{cases}$$

$$= \begin{cases} -\pi_1^*d(\rho_V\omega)\wedge\pi_2^*\eta & \text{on } U\times N \\ \pi_1^*d(\rho_U\omega)\wedge\pi_2^*\eta & \text{on } V\times N \end{cases}$$

$$= \pi_1^*\delta\omega\wedge\pi_2^*\eta. \qquad \square$$